### 9.2 OBJECTIVES

1. Simplify expressions involving numeric radicals
2. Simplify expressions involving algebraic radicals

In Section 9.1, we introduced the radical notation. For most applications, we will want to make sure that all radical expressions are in simplest form. To accomplish this, the following three conditions must be satisfied.

## Rules and Properties: Square Root Expressions in Simplest Form

An expression involving square roots is in simplest form if

1. There are no perfect-square factors in a radical.
2. No fraction appears inside a radical.
3. No radical appears in the denominator.

For instance, considering condition 1 ,
$\sqrt{17}$ is in simplest form because 17 has no perfect-square factors
whereas
$\sqrt{12}$ is not in simplest form
because it does contain a perfect-square factor.
$\sqrt{12}=\sqrt{\Sigma_{\text {A perfect square }}}$
To simplify radical expressions, we'll need to develop two important properties. First, look at the following expressions:

$$
\begin{array}{r}
\sqrt{4 \cdot 9}=\sqrt{36}=6 \\
\sqrt{4} \cdot \sqrt{9}=2 \cdot 3=6
\end{array}
$$

Because this tells us that $\sqrt{4 \cdot 9}=\sqrt{4} \cdot \sqrt{9}$, the following general rule for radicals is suggested.

## Rules and Properties: Property 1 of Radicals

For any positive real numbers $a$ and $b$,
$\sqrt{a b}=\sqrt{a} \cdot \sqrt{b}$
In words, the square root of a product is the product of the square roots.

NOTE Perfect-square factors are $1,4,9,16,25,36,49,64,81$, 100, and so on.

NOTE Apply Property 1.
NOTE Notice that we have removed the perfect-square factor from inside the radical, so the expression is in simplest form.
NOTE It would not have helped to write
$\sqrt{45}=\sqrt{15 \cdot 3}$
because neither factor is a perfect square.
NOTE We look for the largest perfect-square factor, here 36 .

NOTE Then apply Property 1.

Let's see how this property is applied in simplifying expressions when radicals are involved.

## Example 1

## Simplifying Radical Expressions

Simplify each expression.
(a) $\sqrt{12}=\sqrt{4 \cdot 3}$

A perfect square

$$
\begin{aligned}
& =\sqrt{4} \cdot \sqrt{3} \\
& =2 \sqrt{3}
\end{aligned}
$$

(b) $\sqrt{45}=\sqrt{9 \cdot 5}$

A perfect square
$=\sqrt{9} \cdot \sqrt{5}$

$$
=3 \sqrt{5}
$$

(c) $\sqrt{72}=\sqrt{36 \cdot 2}$

A perfect square
$=\sqrt{36} \cdot \sqrt{2}$
$=6 \sqrt{2}$
(d) $5 \sqrt{18}=5 \sqrt{9 \cdot 2}$

A perfect square

$$
=5 \cdot \sqrt{9} \cdot \sqrt{2}=5 \cdot 3 \cdot \sqrt{2}=15 \sqrt{2}
$$

Be Careful! Even though
$\sqrt{a \cdot b}=\sqrt{a} \cdot \sqrt{b}$
$\sqrt{a+b} \quad$ is not the same as $\quad \sqrt{a}+\sqrt{b}$
Let $a=4$ and $b=9$, and substitute.
$\sqrt{a+b}=\sqrt{4+9}=\sqrt{13}$
$\sqrt{a}+\sqrt{b}=\sqrt{4}+\sqrt{9}=2+3=5$
Because $\sqrt{13} \neq 5$, we see that the expressions $\sqrt{a+b}$ and $\sqrt{a}+\sqrt{b}$ are not in general the same.

## CHECK YOURSELF 1

Simplify.
(a) $\sqrt{20}$
(b) $\sqrt{75}$
(c) $\sqrt{98}$
(d) $\sqrt{48}$

NOTE By our first rule for radicals.
NOTE $\sqrt{x^{2}}=x$ (as long as $x$ is positive).

The process is the same if variables are involved in a radical expression. In our remaining work with radicals, we will assume that all variables represent positive real numbers.

## Example 2

## Simplifying Radical Expressions

Simplify each of the following radicals.
(a) $\sqrt{x^{3}}=\sqrt{x^{2} \cdot x}$ A perfect square

$$
\begin{aligned}
& =\sqrt{x^{2}} \cdot \sqrt{x} \\
& =x \sqrt{x}
\end{aligned}
$$

(b) $\sqrt{4 b^{3}}=\sqrt{4 \cdot b^{2} \cdot b}$

Perfect squares

$$
\begin{aligned}
& =\sqrt{4 b^{2}} \cdot \sqrt{b} \\
& =2 b \sqrt{b}
\end{aligned}
$$

NOTE Notice that we want the perfect-square factor to have the largest possible even exponent, here 4 . Keep in mind that
$a^{2} \cdot a^{2}=a^{4}$
(c) $\sqrt{18 a^{5}}=\sqrt{9 \cdot a^{4} \cdot 2 a}$

Perfect squares
$=\sqrt{9 a^{4}} \cdot \sqrt{2 a}$

$$
=3 a^{2} \sqrt{2 a}
$$

## CHECK YOURSELF 2

Simplify.
(a) $\sqrt{9 x^{3}}$
(b) $\sqrt{27 m^{3}}$
(c) $\sqrt{50 b^{5}}$

To develop a second property for radicals, look at the following expressions:

$$
\begin{aligned}
& \sqrt{\frac{16}{4}}=\sqrt{4}=2 \\
& \frac{\sqrt{16}}{\sqrt{4}}=\frac{4}{2}=2
\end{aligned}
$$

Because $\sqrt{\frac{16}{4}}=\frac{\sqrt{16}}{\sqrt{4}}$, a second general rule for radicals is suggested.

NOTE Apply Property 2 to write the numerator and denominator as separate radicals.

NOTE Apply Property 2.

NOTE Apply Property 2.

NOTE Factor $8 x^{2}$ as $4 x^{2} \cdot 2$.

NOTE Apply Property 1 in the numerator.

## Rules and Properties: Property 2 of Radicals

For any positive real numbers $a$ and $b$,
$\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$
In words, the square root of a quotient is the quotient of the square roots.

This property is used in a fashion similar to Property 1 in simplifying radical expressions. Remember that our second condition for a radical expression to be in simplest form states that no fraction should appear inside a radical. Example 3 illustrates how expressions that violate that condition are simplified.

## Example 3

## Simplifying Radical Expressions

Write each expression in simplest form.
(a) $\sqrt{\frac{9}{4}}=\frac{\sqrt{9}}{\sqrt{4}} \quad\left\{\begin{array}{l}\text { Remove any } \\ \text { perfect squares } \\ \text { from the radical. }\end{array}\right.$

$$
=\frac{3}{2}
$$

(b) $\sqrt{\frac{2}{25}}=\frac{\sqrt{2}}{\sqrt{25}}$

$$
=\frac{\sqrt{2}}{5}
$$

(c) $\sqrt{\frac{8 x^{2}}{9}}=\frac{\sqrt{8 x^{2}}}{\sqrt{9}}$

$$
\begin{aligned}
& =\frac{\sqrt{4 x^{2} \cdot 2}}{3} \\
& =\frac{\sqrt{4 x^{2}} \cdot \sqrt{2}}{3} \\
& =\frac{2 x \sqrt{2}}{3}
\end{aligned}
$$

## CHECK YOURSELF 3

## Simplify.

(a) $\sqrt{\frac{25}{16}}$
(b) $\sqrt{\frac{7}{9}}$
(c) $\sqrt{\frac{12 x^{2}}{49}}$

NOTE We begin by applying Property 2.

NOTE We can do this because we are multiplying the fraction by $\frac{\sqrt{3}}{\sqrt{3}}$ or 1 , which does not change its value.

## NOTE

$$
\begin{aligned}
& \sqrt{2} \cdot \sqrt{5}=\sqrt{2 \cdot 5}=\sqrt{10} \\
& \sqrt{5} \cdot \sqrt{5}=5
\end{aligned}
$$

In our previous examples, the denominator of the fraction appearing in the radical was a perfect square, and we were able to write each expression in simplest radical form by removing that perfect square from the denominator.

If the denominator of the fraction in the radical is not a perfect square, we can still apply Property 2 of radicals. As we will see in Example 4, the third condition for a radical to be in simplest form is then violated, and a new technique is necessary.

## Example 4

## Simplifying Radical Expressions

Write each expression in simplest form.
(a) $\sqrt{\frac{1}{3}}=\frac{\sqrt{1}}{\sqrt{3}}=\frac{1}{\sqrt{3}}$

Do you see that $\frac{1}{\sqrt{3}}$ is still not in simplest form because of the radical in the denominator?
To solve this problem, we multiply the numerator and denominator by $\sqrt{3}$. Note that the denominator will become
$\sqrt{3} \cdot \sqrt{3}=\sqrt{9}=3$
We then have
$\frac{1}{\sqrt{3}}=\frac{1 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}=\frac{\sqrt{3}}{3}$

The expression $\frac{\sqrt{3}}{3}$ is now in simplest form because all three of our conditions are satisfied.
(b) $\sqrt{\frac{2}{5}}=\frac{\sqrt{2}}{\sqrt{5}}$

$$
\begin{aligned}
& =\frac{\sqrt{2} \cdot \sqrt{5}}{\sqrt{5} \cdot \sqrt{5}} \\
& =\frac{\sqrt{10}}{5}
\end{aligned}
$$

and the expression is in simplest form because again our three conditions are satisfied.
(c) $\sqrt{\frac{3 x}{7}}=\frac{\sqrt{3 x}}{\sqrt{7}}$

$$
\begin{aligned}
& =\frac{\sqrt{3 x} \cdot \sqrt{7}}{\sqrt{7} \cdot \sqrt{7}} \\
& =\frac{\sqrt{21 x}}{7}
\end{aligned}
$$

The expression is in simplest form.
(a) $\sqrt{\frac{1}{2}}$
(b) $\sqrt{\frac{2}{3}}$
(c) $\sqrt{\frac{2 y}{5}}$

Both of the properties of radicals given in this section are true for cube roots, fourth roots, and so on. Here we have limited ourselves to simplifying expressions involving square roots.

乐 CHECK YOURSELF ANSWERS

1. (a) $2 \sqrt{5}$; (b) $5 \sqrt{3}$; (c) $7 \sqrt{2}$; (d) $4 \sqrt{3}$
2. (a) $3 x \sqrt{x}$; (b) $3 m \sqrt{3 m}$;
(c) $5 b^{2} \sqrt{2 b}$
3. (a) $\frac{5}{4}$; (b) $\frac{\sqrt{7}}{3}$; (c) $\frac{2 x \sqrt{3}}{7}$
4. (a) $\frac{\sqrt{2}}{2} ;$ (b) $\frac{\sqrt{6}}{3}$; (c) $\frac{\sqrt{10 y}}{5}$

Section $\qquad$ Date $\qquad$

Use Property 1 to simplify each of the following radical expressions. Assume that all variables represent positive real numbers.

1. $\sqrt{18}$
2. $\sqrt{50}$
3. $\sqrt{28}$
4. $\sqrt{108}$
5. $\sqrt{45}$
6. $\sqrt{80}$
7. $\sqrt{48}$
8. $\sqrt{125}$
9. $\sqrt{200}$
10. $\sqrt{96}$
11. $\sqrt{147}$
12. $\sqrt{300}$
13. $3 \sqrt{12}$
14. $5 \sqrt{24}$
15. $\sqrt{5 x^{2}}$
16. $\sqrt{7 a^{2}}$
17. $\sqrt{3 y^{4}}$
18. $\sqrt{10 x^{6}}$
19. $\sqrt{2 r^{3}}$
20. $\sqrt{5 a^{5}}$
21. $\sqrt{27 b^{2}}$
22. $\sqrt{98 m^{4}}$
23. $\sqrt{24 x^{4}}$
24. $\sqrt{72 x^{3}}$

## ANSWERS

1. 
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## ANSWERS

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44. 
45. 
46. 
47. 
48. 
49. $\sqrt{54 a^{5}}$
50. $\sqrt{200 y^{6}}$
51. $\sqrt{x^{3} y^{2}}$
52. $\sqrt{a^{2} b^{5}}$

Use Property 2 to simplify each of the following radical expressions.
29. $\sqrt{\frac{4}{25}}$
30. $\sqrt{\frac{64}{9}}$
31. $\sqrt{\frac{9}{16}}$
32. $\sqrt{\frac{49}{25}}$
33. $\sqrt{\frac{3}{4}}$
34. $\sqrt{\frac{5}{9}}$
35. $\sqrt{\frac{5}{36}}$
36. $\sqrt{\frac{10}{49}}$

Use the properties for radicals to simplify each of the following expressions. Assume that all variables represent positive real numbers.
37. $\sqrt{\frac{8 a^{2}}{25}}$
38. $\sqrt{\frac{12 y^{2}}{49}}$
39. $\sqrt{\frac{1}{5}}$
40. $\sqrt{\frac{1}{7}}$
41. $\sqrt{\frac{3}{2}}$
42. $\sqrt{\frac{5}{3}}$
43. $\sqrt{\frac{3 a}{5}}$
44. $\sqrt{\frac{2 x}{7}}$
45. $\sqrt{\frac{2 x^{2}}{3}}$
46. $\sqrt{\frac{5 m^{2}}{2}}$
47. $\sqrt{\frac{8 s^{3}}{7}}$
48. $\sqrt{\frac{12 x^{3}}{5}}$

Decide whether each of the following is already written in simplest form. If it is not, explain what needs to be done.

49. $\sqrt{10 m n}$
50. $\sqrt{18 a b}$
51. $\sqrt{\frac{98 x^{2} y}{7 x}}$
52. $\frac{\sqrt{6 x y}}{3 x}$
53. Find the area and perimeter of this square:


One of these measures, the area, is a rational number, and the other, the perimeter, is an irrational number. Explain how this happened. Will the area always be a rational number? Explain.

54. (a) Evaluate the three expressions $\frac{n^{2}-1}{2}, n, \frac{n^{2}+1}{2}$ using odd values of $n$ :
$1,3,5,7$, etc. Make a chart like the one below and complete it.

(b) Check for each of these sets of three numbers to see if this statement is true: $\sqrt{a^{2}+b^{2}}=\sqrt{c^{2}}$. For how many of your sets of three did this work? Sets of three numbers for which this statement is true are called "Pythagorean triples" because $a^{2}+b^{2}=c^{2}$. Can the radical equation be written in this way:
$\sqrt{a^{2}+b^{2}}=a+b$ ? Explain your answer.
49.
50.
51.
52.

53.
54.


## ANSWERS

a.
b.
c.
d.
e.
f.
g.
h.

## Getting Ready for Section 9.3 [Section 1.6]

Use the distributive property to combine the like terms in each of the following expressions.
(a) $5 x+6 x$
(b) $8 a-3 a$
(c) $10 y-12 y$
(d) $7 m+10 m$
(e) $9 a+7 a-12 a$
(f) $5 s-8 s+4 s$
(g) $12 m+3 n-6 m$
(h) $8 x+5 y-4 x$

## Answers

1. $3 \sqrt{2}$
2. $2 \sqrt{7}$
3. $3 \sqrt{5}$
4. $4 \sqrt{3}$
5. $10 \sqrt{2}$
6. $7 \sqrt{3}$
7. $6 \sqrt{3}$
8. $x \sqrt{5}$
9. $y^{2} \sqrt{3}$
10. $r \sqrt{2 r}$
11. $3 b \sqrt{3}$
12. $2 x^{2} \sqrt{6}$
13. $3 a^{2} \sqrt{6 a}$
14. $x y \sqrt{x}$
15. $\frac{2}{5}$
16. $\frac{3}{4}$
17. $\frac{\sqrt{3}}{2}$
18. $\frac{\sqrt{5}}{6}$
19. $\frac{2 a \sqrt{2}}{5}$
20. $\frac{\sqrt{5}}{5}$
21. $\frac{\sqrt{6}}{2}$
22. $\frac{\sqrt{15 a}}{5}$
23. $\frac{x \sqrt{6}}{3}$
24. $\frac{2 s \sqrt{14 s}}{7}$
25. Simplest form
26. Remove the perfect-square factors from the radical and simplify.
27. 

a. $11 x$
b. $5 a$
c. $-2 y$
d. 17 m
e. $4 a$
f. $s$
g. $6 m+3 n$
h. $4 x+5 y$

