7.3 Integral and Rational Zeros of Polynomials

Integral Zeros

Integral Zeros Theorem: if an integer a is a zero of a polynomial function with integral coefficients and a leading coefficient of 1, then a is a factor of the constant term of the polynomial.

Consider: \( f(x) = x^3 + 4x^2 - 7x - 10 \)

2 → zero of \( f(x) \)

factor of constant term \(-10\)
Example 1: Determine the integral zeros of \( f(x) = x^3 - 6x^2 + 3x + 10 \) and write \( f(x) \) as the product of linear factors.

\[
\frac{(x-a)}{\text{constant: 10}}
\]

possible int. zeros: \( \pm 1, \pm 2, \pm 5, \pm 10 \)

Use Syn. Div to test.

\[
\begin{array}{c|cccc}
 & 1 & -6 & -3 & 10 \\
 1 & & 1 & -5 & -2 & 8 \\
 \hline
 & 1 & -5 & -2 & 8 & 8
\end{array}
\]

\[
\begin{array}{c|cccc}
 & 1 & -6 & 3 & 10 \\
 -1 & & 1 & 7 & 10 & \text{Our yes!} \\
 \hline
 & 1 & -7 & 10 & 0 & \text{Our yes!}
\end{array}
\]

\[
(x+1)(x^2 - 7x + 10)
\]

\[
(x+1)(x-5)(x-2)
\]
Example 2: Determine the integral zeros of $f(x) = x^3 + 3x^2 - 2x - 6$ and write $f(x)$ as the product of linear factors.

Possible zeros: $\pm 1, \pm 2, \pm 3, \pm 6$

\[
\begin{array}{c|cccc}
-3 & 1 & 3 & -2 & -6 \\
 & -3 & 0 & 6 & \downarrow \checkmark \Rightarrow R = 0 ! \\
\hline
1 & 1 & 0 & -2 & 0 & \leq
\end{array}
\]

\[(x+3)(x^2-2)\]

\[(x+3)(x+\sqrt{2})(x-\sqrt{2}) = f(x)\]
Rational Zeros Theorem: Let \( f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + ... + a_1 x + a_0, a_0 \neq 0 \), be a polynomial function in standard form that has integral coefficients. Then if the nonzero rational number \( \frac{p}{q} \) in lowest terms is a zero of \( p(x) \), \( p \) must be a factor of the constant \( a_0 \) and \( q \) must be a factor of the leading coefficient \( a_n \).

Consider \( f(x) = 4x^3 + 7x^2 - 43x + 35 \)
Example 3: Determine the rational zeros of \( f(x) = 12x^3 - 16x^2 - 5x + 3 \) and write \( f(x) \) as the product of linear factors.

\[ 3^{(p)}: \pm 1, \pm 3 \]

\[ 12^{(q)}: \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12 \]

Possible rational zeros:

\[ \frac{p}{q} : \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \pm \frac{1}{6}, \pm \frac{1}{12}, \pm 3, \pm \frac{3}{2}, \pm \frac{3}{4} \]

\[ -\frac{1}{2} \begin{array}{c} 12 \end{array} \]
\[ \begin{array}{rrrr} -16 & -5 & 3 \\ -6 & -11 & -3 \end{array} \]

\[ 12 - 22 \quad \text{no roots} \]

\( (x + \frac{1}{2})(12x^2 - 22x + 6) \)

\( (x + \frac{1}{2})(3x - 1)(2x - 3) = 0 \)
Example 4: Determine the rational zeros of \( f(x) = 2x^4 + 5x^3 - 5x - 2 \) and write \( f(x) \) as the product of linear factors.